$$\frac{\theta}{2} = t$$

in second integral, we get,

$$d\theta = 2dt \Rightarrow \int_0^{\frac{\pi}{2}} \sec^2\left(\frac{\theta}{2}\right) d\theta - \frac{1}{2}\int_0^{\frac{\pi}{4}} (1+\tan^2(t))(\sec^2(t))dt$$

$$\Rightarrow \left[\tan\left(\frac{\theta}{2}\right)\right]_0^{\frac{\pi}{2}} - \frac{1}{2}\left[\int_0^{\frac{\pi}{4}} \sec^2(t)dt + \int_0^{\frac{\pi}{4}} (\tan^2(t))(\sec^2(t))dt\right]$$

$$\Rightarrow 1 - \frac{1}{2}\left[\left[\tan(t)\right]_0^{\frac{\pi}{4}} + \left[\frac{\tan^3(t)}{3}\right]_0^{\frac{\pi}{4}}\right]$$

$$\Rightarrow 1 - \frac{1}{2}\left(1+\frac{1}{3}\right) \Rightarrow 1 - \frac{2}{3}$$

(OR)

$$I = \frac{1}{3}$$

(Answer)

Shivam Sharma

Fourth solution. We have

$$I := \int_0^{\pi/2} \frac{\sin x}{\left(1+\sqrt{\sin 2x}\right)^2} dx = [x = \pi/4 - \theta; dx = -d\theta] =$$

$$= \int_{-\pi/4}^{\pi/4} \frac{\sin(\pi/4-\theta)}{\left(1+\sqrt{\sin(\pi/2-2\theta)}\right)^2} d\theta = \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} \frac{\cos\theta - \sin\theta}{\left(1+\sqrt{\cos 2\theta}\right)^2} d\theta =$$

$$= \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} \frac{\cos\theta}{\left(1+\sqrt{\cos 2\theta}\right)^2} d\theta - \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} \frac{\sin\theta}{\left(1+\sqrt{\cos 2\theta}\right)^2} d\theta.$$

Since function $\dfrac{\sin\theta}{\left(1+\sqrt{\cos 2\theta}\right)^2}$ is odd and function $\dfrac{\cos\theta}{\left(1+\sqrt{\cos 2\theta}\right)^2}$ is even on $[-\pi/4, \pi/4]$ then

$$\int_{-\pi/4}^{\pi/4} \frac{\sin\theta}{\left(1+\sqrt{\cos 2\theta}\right)^2} d\theta = 0$$

and

$$\int_{-\pi/4}^{\pi/4} \frac{\cos\theta}{\left(1+\sqrt{\cos 2\theta}\right)^2} d\theta = 2\int_{0}^{\pi/4} \frac{\cos\theta}{\left(1+\sqrt{\cos 2\theta}\right)^2} d\theta.$$

Thus,

$$I = \sqrt{2}\int_{0}^{\pi/4} \frac{\cos\theta}{\left(1+\sqrt{\cos 2\theta}\right)^2} d\theta = \int_{0}^{\pi/4} \frac{\sqrt{2}\cos\theta}{\left(1+\sqrt{1-2\sin^2\theta}\right)^2} d\theta =$$

$$= \left[s := \sqrt{2}\sin\theta; ds = \left(\sqrt{2}\cos\theta\right) d\theta\right] = \int_0^1 \frac{ds}{\left(1+\sqrt{1-s^2}\right)^2}.$$

For calculation of the latter integral we will use substitution $s = \dfrac{2t}{1+t^2}$ $\left(t \mapsto \dfrac{2t}{1+t^2} : [0,1] \longrightarrow [0,1]\right.$ is bijection).

Then

$$ds = \frac{2\left(1-t^2\right)}{\left(1+t^2\right)^2} dt, \sqrt{1-s^2} = \frac{1-t^2}{1+t^2}, 1+\sqrt{1-s^2} = \frac{2}{1+t^2}$$

and therefore,

$$\int_0^1 \frac{ds}{\left(1+\sqrt{1-s^2}\right)^2} = \int_0^1 \frac{\dfrac{2\left(1-t^2\right)}{\left(1+t^2\right)^2}}{\left(\dfrac{2}{1+t^2}\right)^2} dt = \frac{1}{2}\int_0^1 \left(1-t^2\right) dt =$$

$$= \frac{1}{2}\left(t - \frac{t^3}{3}\right)_0^1 = \frac{1}{2}\cdot\frac{2}{3} = \frac{1}{3}.$$

Arkady Alt